Math 601 Final (sample test)

Name: ______________________

This exam has 11 questions, for a total of 150 points.

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	10	
2	10	
3	15	
4	10	
5	10	
6	15	
7	20	
8	20	
9	15	
10	15	
11	10	
Total:	150	

Question 1. (10 pts)

(a) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}$. The eigenvalues of A are 1, 2 and 3. Is A diagonalizable? Justify your answer.

(b) Suppose $F : \mathbb{R}^4 \to \mathbb{R}^7$ is linear mapping. Can F be surjective? Justify your answer.

Question 2. (10 pts)

Determine whether

$$f(z) = e^{-y}((x+1)\cos x - y\sin(x)) + ie^{-y}((x+1)\sin x + y\cos x)$$

is analytic on $\mathbb{C}$, where $z = x + iy$.

Question 3. (15 pts)

Evaluate the integral

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz$$

where C is the circle centered at 0 with radius 4.

Question 4. (10 pts)

Evaluate

$$\int_\gamma (3z^2 + 1)dz$$

where γ is the curve $\gamma(t) = (\sin t, t^2 + t)$ for $t \in [0, \pi]$, that is, the curve starts at $(0, 0)$ and ends at $(0, \pi^2 + \pi)$.

Question 5. (10 pts)

Use the residue theorem to evaluate the following integral

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}$$

Question 6. (15 pts)

Use the residue theorem to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)} dx$$

Question 7. (20 pts)

Given the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Find all eigenvalues of A.

(b) Find a basis for each eigenspace.

(c) Determine whether A is diagonalizable. If yes, find an invertible matrix S so that

$$S^{-1}AS$$

is diagonal. If not, explain why.

Question 8. (20 pts)

Let V be the subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 4 \end{bmatrix}.$$

(a) Use Gram-Schmidt process to find an orthonormal basis of V.

(b) Find the projection of

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

onto V.

Question 9. (15 pts)

Let V be the vector space spanned by $\{e^x, e^{-x}, xe^x, xe^{-x}\}$. Accept as a fact that

$$e^x,\ e^{-x},\ xe^x,\ xe^{-x}$$

form a basis for V. Let us denote this basis by $\mathfrak{B}$. Let

$$T(f) = 2f - f'$$

be a linear transformation from V to V .

(a) Find the $\mathfrak{B}$-matrix of T.

(b) Is T an isomorphism?

Question 10. (15 pts)
Given

$$A = \begin{bmatrix} 2 & 2 & -3 & 1 & 13 \\ 1 & 1 & 1 & 1 & -1 \\ 3 & 3 & -5 & 0 & 14 \\ 6 & 6 & -2 & 4 & 16 \end{bmatrix}$$

(a) Find a basis of $\mathrm{Ker}(A)$.

(b) Find a basis of the row space of A.

Question 11. (10 pts)

Suppose λ is an eigenvalue of an $n \times n$-matrix A.

(a) Show that λ^n is an eigenvalue of A^n.

(b) Consider the matrix

$$B = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I_n$$

where c_i are real numbers. Show that the real number

$$\mu = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0$$

an eigenvalue of B.